Order Theory of Functional Equations

for static cost analysis, and beyond

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Introduction

Functional equations arise in many places, but **cannot always be solved.**

They are important objects in static analysis (**"the art of automatically bounding the behaviour of systems"**), and other areas of science, where they can describe many systems, and encode many problems.

- Imperative and declarative programs can be given meaning by (functional) **semantic equations** (consider recursion, loops...),
- **Resource consumption of a program** may be expressed as the solution of a recurrence equation,
- Complex **differential equations** arise in many areas, including cyberphysical systems, biochemical reactions networks...

Problem: bound the solutions of numerical functional equations.
Equivalently: produce information on **numerical functions** constructed recursively **by operators** $\Phi \in \big((\mathcal{D} \to L) \to (\mathcal{D} \to L)\big)$.

➔ We combine insights from **functional numerical equations** and **(order-theoretical) fixpoint equations**.

Equations as Operators

Example

Search $f : \mathbb{N}^2 \to \mathbb{R}$ such that

$$f(n,c) = \begin{cases} f(n-1,0)+n+300 & \text{if } n>0 \text{ and } c \geq 100, \\ f(n-1,c+1)+n & \text{if } n>0 \text{ and } c<100, \\ c & \text{if } n=0, \end{cases}$$

➔

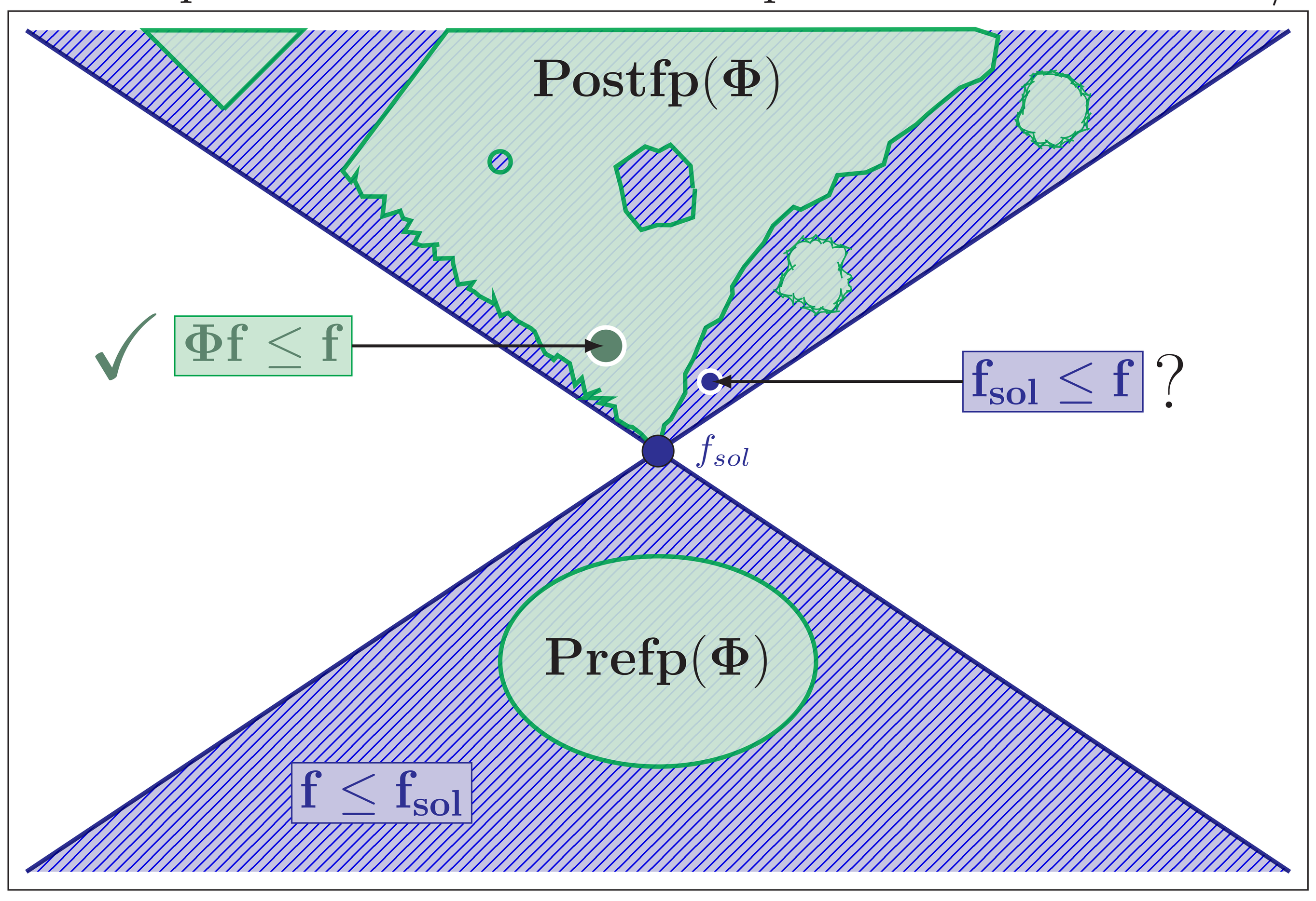
Search $f : \mathbb{N}^2 \to \mathbb{R}$ such that $f = \Phi f$, i.e. $f \in \mathrm{Fixp}(\Phi)$, where

$$\Phi : (\mathbb{N}^2 \to \mathbb{R}) \to (\mathbb{N}^2 \to \mathbb{R})$$

$$f \mapsto (n,c) \mapsto \begin{cases} f(n-1,0)+n+300 & \text{if } n>0 \text{ and } c \geq 100, \\ f(n-1,c+1)+n & \text{if } n>0 \text{ and } c<100, \\ c & \text{if } n=0. \end{cases}$$

Order in function space

Equations ↔ Operators Solutions ↔ Fixpoints Bounds ← Pre/Postfixp



Take a function space $\mathcal{D} \to L$, and order it pointwise: $f \dot{\leq} g \overset{\Delta}{\Leftrightarrow} \forall x, f(x) \leq g(x)$.
Two flavours: purely numerical (e.g. $L = (\overline{\mathbb{R}}, \leq)$) or set-based (e.g. $L = (\mathcal{I}(\mathbb{R}), \sqsubseteq)$).

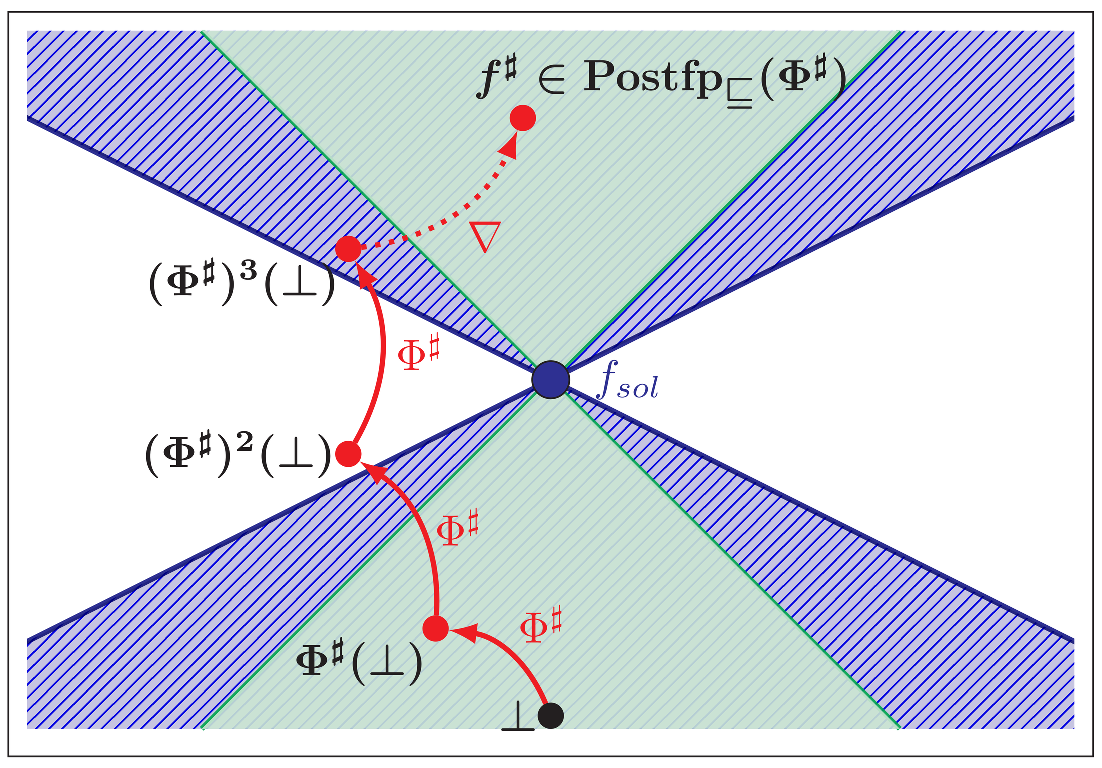
Theorem: A Knaster-Tarski corollary

Let $\Phi : (\mathcal{D} \to L) \to (\mathcal{D} \to L)$ be a **monotone equation** (i.e. $f \leq g \Rightarrow \Phi f \leq \Phi g$).

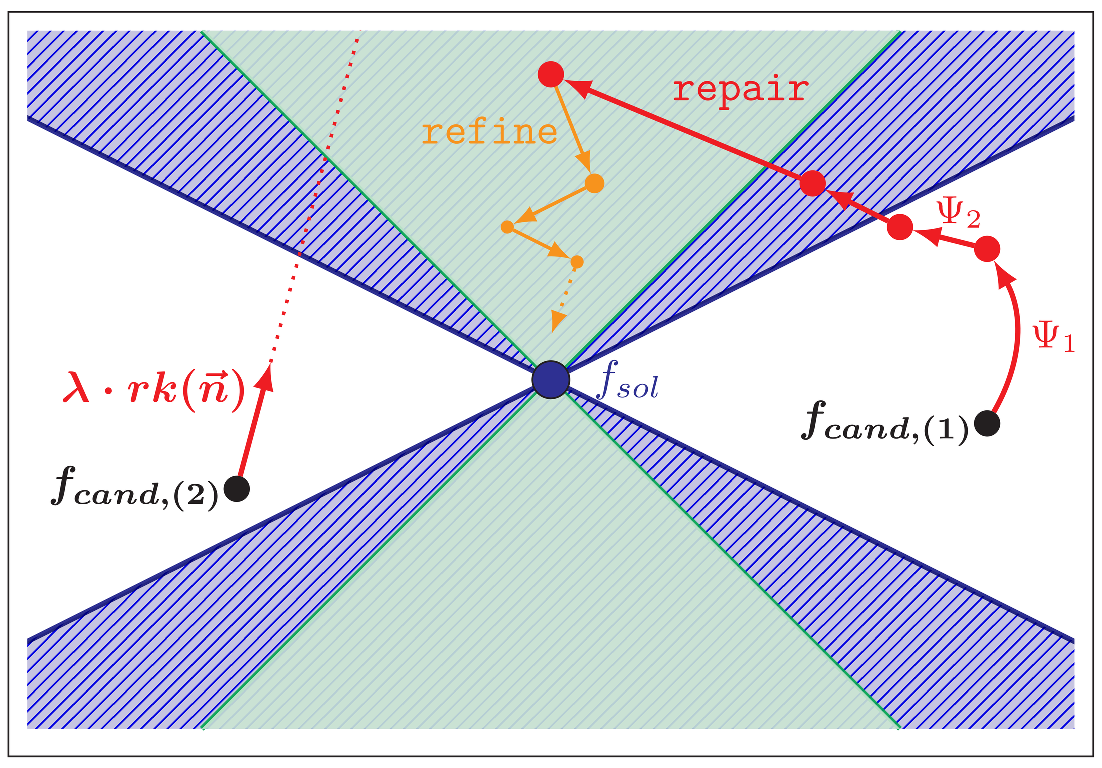
- If $f \in \mathrm{Postfp}(\Phi)$, i.e. $\Phi f \leq f$, then $\mathrm{lfp}\, \Phi \leq f$.
- If $f \in \mathrm{Prefp}(\Phi)$, i.e. $f \leq \Phi f$, then $f \leq \mathrm{gfp}\, \Phi$.

When the equation *terminates unconditionally*, $\mathrm{lfp}\, \Phi = \mathrm{gfp}\, \Phi =: f_{sol}$.

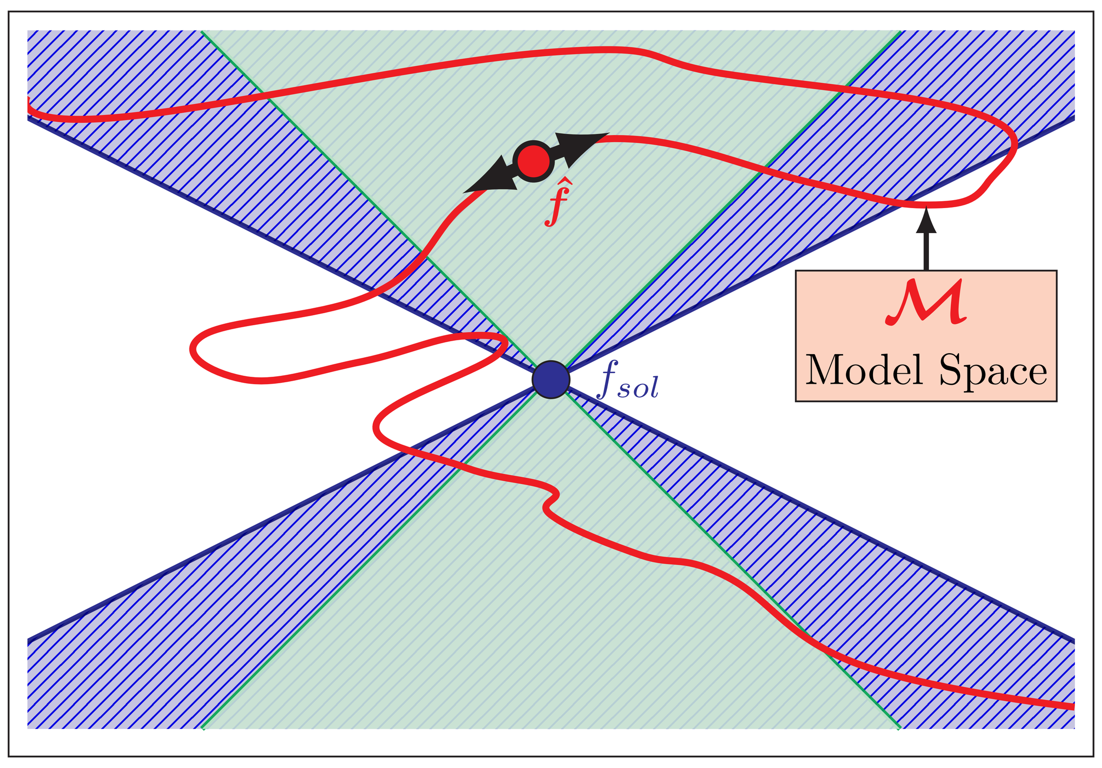
Space exploration: examples of pre/postfixpoint search strategies



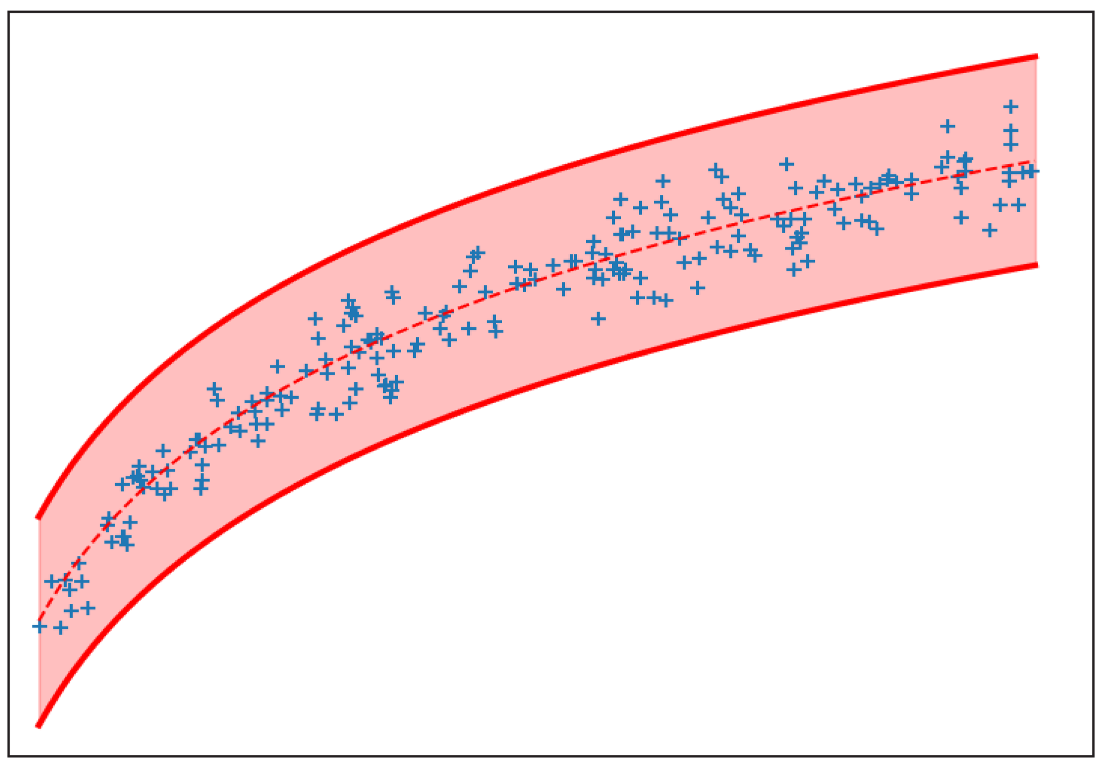
Abstract Interpretation



Geometry-based expression **Repair**

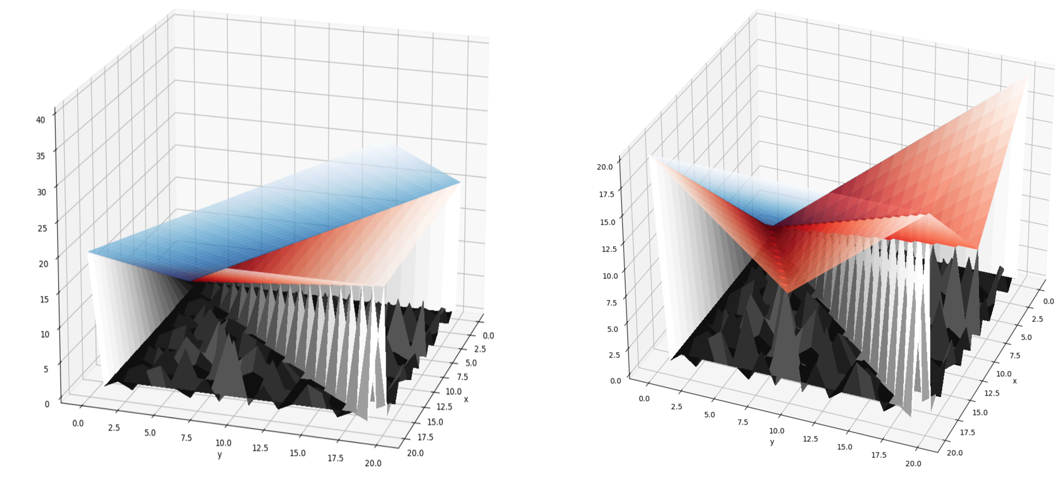


Search on subvarieties: **Templates**, ∀-elim



Constrained **Optimisation**, with *provability constraints*

Ex.: simple bound on a gcd program



Candidate $\hat{f}$, iterate $\Phi\hat{f}$, unkown solution f_{sol}.

Some function space abstractions

Symbolic functions	Numerical-set functions	Abstract num. set functions	Numerical functions	"Flowpipes" Simple bounds

$\Phi_{\mathtt{Data}}$ ⇄ (size abstraction) $\Phi_{\mathcal{P}}$ ⇄ (codomain abstraction) $\Phi_{\mathcal{I}}$ ⇢ (separability (≤-monotonicity), reductions, ...) $\Phi_{\mathbb{N}}$ ⇄ ($\mathfrak{B}$-bounds abstraction) $\Phi^\sharp$

$\mathtt{Data} \to \mathcal{P}(\mathtt{Data} \times \mathbb{R})$ $\mathbb{N}^r \to \mathcal{P}(\mathbb{N}^s \times \mathbb{R})$ $\mathbb{N}^r \to \mathcal{I}(\mathbb{N})^s \times \mathcal{I}(\mathbb{R})$ $\mathbb{N}^k \to \overline{\mathbb{R}}$ $D^\sharp$

Supported equations

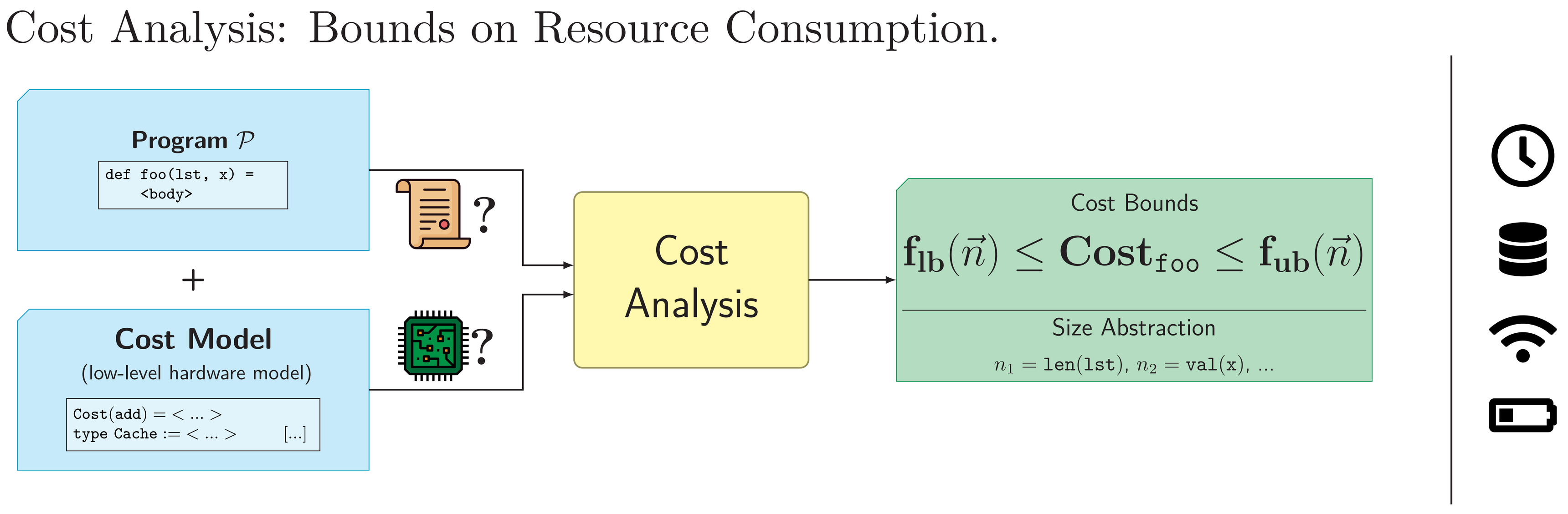
Flexible. Supports affine equations with $a_{i,j} \geq 0$

$$\Phi(f)(\vec{n}) = \begin{cases} \cdots \\ \sum_{j=1}^{k_i} \big(a_{i,j}(\vec{n}) \cdot f(\phi_{i,j}(\vec{n}))\big) + b_i(\vec{n}) & \text{if } \varphi_i(\vec{n}) \\ \cdots, \end{cases}$$

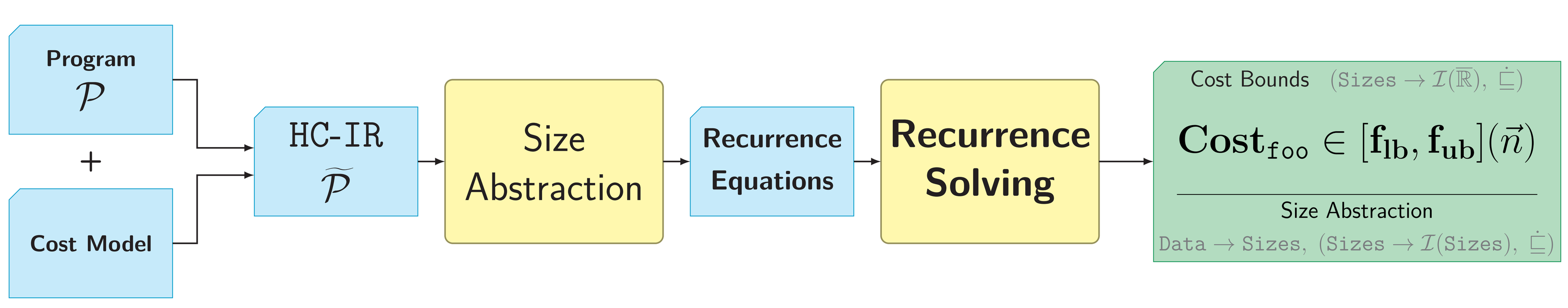
min/max of equations, unbounded optimisation, some nested calls, ... We can obtain **piecewise**, **non-linear** bounds.

Origin and Motivation: Cost Analysis

Cost Analysis: Bounds on Resource Consumption.



Pipeline implemented in CiaoPP (and other analysers).



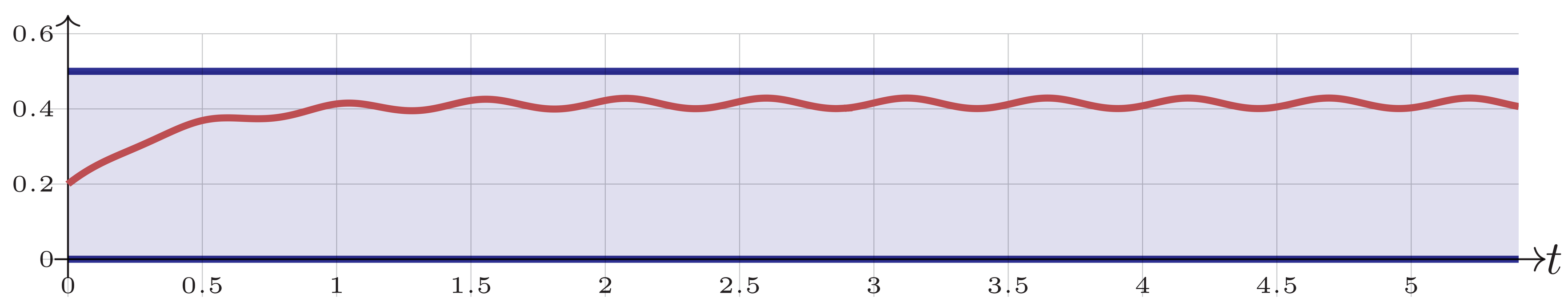
Continuous Systems

Non-standard analysis with infinitesimal ϵ, within **hyperreals** $^*\mathbb{R}$?

$$v : \mathbb{R}_+ \to \mathbb{R}, \qquad v(0) = v_0, \qquad \dot{v} = -\alpha \cdot v^2 - \beta \cdot v + \gamma$$

$$\Phi \in \mathrm{End}\big({}^*\mathbb{R}_+ \to \mathcal{I}({}^*\mathbb{R}), \dot{\sqsubseteq}_{\mathcal{I}}\big)$$
$$:= f \mapsto t \mapsto \begin{cases} [v_0, v_0] & \text{if } t < \epsilon \\ f(t-\epsilon) - \alpha \cdot \epsilon \cdot f(t-\epsilon) \cdot_{\mathcal{I}} f(t-\epsilon) - \beta \cdot \epsilon \cdot f(t-\epsilon) + \epsilon \cdot \gamma & \text{if } t \geq \epsilon \end{cases}$$

For $\hat{f} : t \mapsto [0, M]$, $\Phi\hat{f} \dot{\sqsubseteq}_{\mathcal{I}} \hat{f}$ whenever $x_0 \in [0, M]$, $\alpha, \beta, \gamma \geq 0$ and $M \geq \gamma/\beta$.



Example with $\alpha(t) \in [0, 2]$, $\beta = 2$, $\gamma = 1$, $v_0 = 0.2$.